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Consolidation of multilayered half space with anisotropic permeability and compressible constituents

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Abstract

This paper presents analytical solutions of consolidation for poroelastic and multilayered half space, and anisotropy of the permeability and the compressibility of the pore fluid are considered. State vector method together with Laplace–Hankel transform techniques are used to solve the basic governing equations, and obtain the transfer matrix in a clearly arranged way. Forward and backward transfer matrix techniques are utilized in the analytical formulation of solutions for the multilayered half space. A numerical inversion scheme of Crump's method is adopted to obtain time-domain solution. Numerical results are presented for a single homogeneous soil layer and a multilayered half space, and they illustrate the influences of the anisotropy of permeability and the compressibility of the pore fluid on the consolidation of the soils.

Keywords: State vector method; Laplace–Hankel transform; Consolidation; Anisotropy; Compressible; Multilayered

1. Introduction

The stress-induced flow of interstitial fluid in porous media has been suggested as accounting for a variety of phenomena encountered in geomechanics. The general theory of a porous media taking into account the coupling between the solid and fluid stresses and strains is presented in the pioneering work of Biot (1941).

In many cases, natural soils have been created through a sedimentation process that determines a typical soil fabric in which horizontal stratification planes can be distinguished. This process gives the soil mass a marked degree of anisotropy so that the permeability is different in horizontal and vertical directions. Soil layers which are created in different geological conditions may have apparently different behavior of poroelasticity and permeability. Therefore it is necessary to take into account the anisotropy of permeability and layering behavior of the media.

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Compressibility of the pore fluid is important, and it is considered in this paper under the following two cases of consolidation:

(1) *Consolidation of soil with high saturation degree*. Many practical problems are involved with the consolidation of partially saturated soil, such as earth dam and embankment built of three-phase compacted clay, and landfills including or producing gas (Wentz, 1989). An important and frequently encountered special case is that in which the degree of saturation is high (approximately more than 70%) so that the liquid phase becomes continuous while the gas phase becomes discontinuous and occluded in the form of bubbles in the liquid phase, and the surface tension maintains the differential pressure between pore gas and pore water pressure (Pietruszczzak and Pande, 1996). As the saturation degree is further increased the bubbles and pore water behave as a ''homogeneous compressible fluid'' flowing under the pore water pressure gradient, surface tension effect appears to be unimportant, and this condition prevails at degree of saturation greater than about 85% (Koning, 1963; Sparks, 1963; Schuurman, 1966; Chang and Duncan, 1983; Okusa, 1985).

With neglect of gas solubility in water which appears to be justified, the compressibility of the pore fluid mixture β is given by (Pietruszczzak and Pande, 1996)

$$\beta = \frac{1}{K} = \frac{S_r}{K_w} + \frac{1 - S_r}{P_{a0} - \frac{T}{3\rho_v}\frac{S_r}{\sqrt{1-S_r}}}, \quad S_r > 70\%, \tag{1a}$$

where K and K_w are the average bulk moduli of pore fluid mixture and air free water, respectively, S_r is pore water saturation degree, P_{a0} is the total gas pressure, T is the surface tension force, and ρ_v is the average pore size.

By ignoring the effect of surface tension ($P_{a0} = P_{w0}$), the compressibility of the mixture is approximately represented by (Koning, 1963; Schuurman, 1966; Fredlund, 1976; Okusa, 1985)

$$\beta = \frac{1}{K} = \frac{S_r}{K_w} + \frac{1 - S_r}{P_{w0}}, \quad S_r > 85\%, \tag{1b}$$

where P_{w0} is absolute pore water pressure.

A simple analysis by Verruijt (1969) indicates an upper bound for the compressibility of the pore fluid mixture

$$\beta = \frac{1}{K} = \frac{1}{K_w} + \frac{1 - S_r}{P_{w0}}, \quad 1 - S_r \ll 1, \tag{1c}$$

this condition prevails at degree of saturation with $1 - S_r \ll 1$, and it will be applied in our following analysis. This condition has been applied by many other researchers (e.g. Madsen, 1978; Yamamoto et al., 1978; Jeng and Seymour, 1997; Yang and Sato, 2001). From Eq. (1c), it can readily be shown that even a very small amount of gas in soil will dramastically reduce the bulk modulus of fluid.

The importance of pore fluid compressibility for consolidation problem has been demonstrated by many researchers. Cheng and Liggett (1984) concluded that compressibility of pore fluid would drastically alter the soil behavior in both the consolidation process and the pore pressure distribution. Booker and Carter (1987) demonstrated that the compressibility of the pore fluid can have a significant influence on the rate of consolidation of the soil around the point sink and thus on the settlement of the surface of the half space. Yue et al. (1994) presented that the presence of a compressible pore fluid reduces the generation of excess pore water pressure in the poroelastic seabed layer. Besides, its significance has also been indicated in consolidation problem under ocean wave loading (Madsen, 1978; Okusa, 1985; Jeng and Seymour, 1997) or earthquake excitation (Yang and Sato, 2001).

(2) *Consolidation of saturated porous rock*. For the media of water-saturated rock, pore water is not effectively incompressible. In many cases, the stiffness of the porous rock is much larger than that of the air free water, therefore the compressibility of pore fluid should be considered (Skempton, 1954).

Analytical solutions related to a poroelastic medium consolidation have been obtained by many researchers including Gibson et al. (1970), Booker (1974), Booker and Small (1982a,b, 1987), Vardoulakis and Harnpattanapanich (1986), Harnpattanapanich and Vardoulakis (1987) and Senjuntichai and Rajapakse (1995). In almost all these investigations, it was assumed that the permeability is isotropic, the pore fluid is incompressible, and the medium has finite thickness. Such assumption made it impossible to analytically examine the roles of anisotropy of permeability and compressibility of pore fluid on the consolidation process, besides, the assumption that the poroelastic medium has finite thickness with completely permeable or impermeable hydraulic base and completely rigid and rough mechanical base is not realistic for most real cases.

In this paper our study will be focused on the development of the analytical solutions for the consolidation of multilayerd poroelastic media with anisotropic permeability, compressible pore fluid ($1 - S_r \ll 1$) and infinite thickness. Firstly, an efficient state vector method (Zhong et al., 1995; Chen et al., 1998; Chen and Zhao, 1999; Chen, 2003) is adopted to re-express the basic governing equations as two matrix ordinary differential equations with respect to two state vectors composed of displacement, stress, pore water pressure and superficial velocity of the pore fluid. By applying Laplace–Hankel transforms to the matrix differential equations and employing Cayley–Hamilton theorem, the matrix equations are solved and transfer matrix between state vectors at different depths (z) is obtained in Laplace–Hankel transform domain. Secondly, forward and backward transfer matrix methods are utilized to get the analytical solutions for a multilayered poroelastic media. Thirdly the inversions of Hankel and Laplace transforms should be performed to obtain the solutions in the physical domain. A numerical inversion scheme of Crump's method is adopted to obtain time-domain solution. Finally, based on the analytical solutions, numerical results are presented to study a single soil layer and examine the influence of anisotropy of permeability and compressibility of pore fluid on consolidation, a multilayered half space is investigated in order to show the efficiency of the present study.

Therefore, the main objectives of this paper can be summarized as follows: (1) to present an efficient formulation for the development of analytical solutions governing the consolidation problem; (2) to extend the analytical modeling of consolidation to include the anisotropy of permeability and compressibility of pore fluid; and (3) to extend the analytical modeling of media with finite thickness to infinite half space.

2. Governing equations and solutions

Consider a homogeneous poroelastic layer with high saturation degree and infinite in horizontal extent. The cylindrical polar coordinate system (r, z, θ) is here used. The load q_z, q_{rz}, $q_{\theta z}$ is applied at depth h below the surface of the layer with thickness H, and the layer extends a further distance $H - h$ below the loading surface (see Fig. 1).

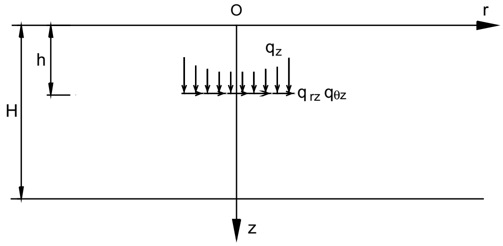

Fig. 1. A homogeneous poroelastic layer subjected to external load.

2.1. Static equilibrium equations

In the absence of increase in body forces, static equations of equilibrium with respect to conventional cylindrical polar coordinate system take the form

$$\left.\begin{array}{l}\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}=0,\\ \frac{\partial\tau_{rz}}{\partial r}+\frac{\tau_{rz}}{r}+\frac{\partial\sigma_z}{\partial z}+\frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta}=0,\\ \frac{\partial\tau_{rz}}{\partial r}+\frac{\tau_{rz}}{r}+\frac{\partial\sigma_z}{\partial z}+\frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta}=0,\end{array}\right\} \tag{2}$$

where σ_r, σ_θ, σ_z are the total stresses taken as positive in tension and τ_{zr}, $\tau_{r\theta}$, $\tau_{z\theta}$ are the shear stresses.

2.2. Constitutive equations

Under the hypothesis that the pore water and pore gas are mixed as a "homogeneous pore fluid", soil with high saturation degree can be regarded as a quasi-two-phase medium, therefore constitutive equations take the form

$$\left.\begin{array}{l}\sigma_r+p=2\mu\left[\frac{\partial u}{\partial r}+\frac{v}{1-2v}\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}+\frac{\partial w}{\partial z}\right)\right],\\ \sigma_\theta+p=2\mu\left[\left(\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}\right)+\frac{v}{1-2v}\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}+\frac{\partial w}{\partial z}\right)\right],\\ \sigma_z+p=2\mu\left[\frac{\partial w}{\partial z}+\frac{v}{1-2v}\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}+\frac{\partial w}{\partial z}\right)\right],\\ \tau_{zr}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right),\qquad \tau_{z\theta}=\mu\left(\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial\theta}\right),\qquad \tau_{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right),\end{array}\right\} \tag{3}$$

where p is the excess average pressure of the "homogeneous pore fluid" taken as positive in compression, u, v, w are the bulk displacement components in the radial, tangential and vertical directions respectively, μ is the shear modulus and v is Poisson's ratio.

2.3. Mass conservation law

For transient flow with different permeability between horizontal and vertical directions, the pore fluid mass conservation equation of a quasi-static porous medium is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z}\right)+n\beta\frac{\partial p}{\partial t}=\frac{1}{\gamma_w}\left[k_r'\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}\right)+k_z'\frac{\partial^2 p}{\partial z^2}\right], \tag{4}$$

where k_z', k_r' denote the vertical and horizontal permeability coefficients, respectively, n is porosity, γ_w is the unit weight of pore water, t represents time.

2.4. Solutions formulation

Define another four variables as

$$\left.\begin{array}{ll}u_v=\frac{1}{r}\left[\frac{\partial(ru)}{\partial r}+\frac{\partial v}{\partial\theta}\right], & u_h=-\frac{1}{r}\left[\frac{\partial(rv)}{\partial r}-\frac{\partial u}{\partial\theta}\right],\\ \tau_{vz}=\frac{1}{r}\left[\frac{\partial(r\tau_{zr})}{\partial r}+\frac{\partial\tau_{z\theta}}{\partial\theta}\right], & \tau_{hz}=-\frac{1}{r}\left[\frac{\partial(r\tau_{z\theta})}{\partial r}-\frac{\partial\tau_{rz}}{\partial\theta}\right].\end{array}\right\} \tag{5}$$

By suitably manipulating Eqs. (2)–(5), and applying Laplace transform, we can obtain the following expressions:

$$\left.\begin{aligned}
&\frac{\partial\tilde{u}_v}{\partial z}=-\nabla_\theta^2\tilde{w}+\frac{1}{\mu}\tilde{\tau}_{vz},\\
&\frac{\partial\tilde{\sigma}_z}{\partial z}=-\tilde{\tau}_{vz},\\
&\frac{\partial\tilde{w}}{\partial z}=\frac{1-2v}{2\mu(1-v)}\tilde{\sigma}_z+\frac{1-2v}{2\mu(1-v)}\tilde{p}-\frac{v}{1-v}\tilde{u}_v,\\
&\frac{\partial\tilde{\tau}_{vz}}{\partial z}=-\nabla_\theta^2\left(\frac{v}{1-v}\tilde{\sigma}_z+\frac{1-2v}{1-v}\tilde{p}+\frac{2\mu}{1-v}\tilde{u}_v\right),\\
&\frac{\partial\tilde{p}}{\partial z}=-\frac{1}{k_z}\tilde{v}_z,\\
&\frac{\partial\tilde{v}_z}{\partial z}=\gamma^2k_z\nabla_\theta^2\tilde{p}-s\left(\frac{1-2v}{2\mu(1-v)}\tilde{\sigma}_z+\frac{1-2v}{2\mu(1-v)}\tilde{p}+\frac{1-2v}{1-v}\tilde{u}_v\right)-sn\beta\tilde{p},
\end{aligned}\right\} \tag{6a}$$

$$\left.\begin{aligned}
&\frac{\partial\tilde{u}_h}{\partial z}=\frac{2}{f}\tilde{\tau}_{hz},\\
&\frac{\partial\tilde{\tau}_{hz}}{\partial z}=-\frac{a-b}{2}\nabla_\theta^2\tilde{u}_h,
\end{aligned}\right\} \tag{6b}$$

where an overbar "∼" is hereafter used to denote the Laplace transform of a given variable, s is the Laplace variable, $\nabla_\theta^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, $k_r=k'_r/\gamma_w$, $k_z=k'_z/\gamma_w$ and $\gamma^2=k_r/k_z$.

Hereafter we expand the following variables into the Fourier series as:

$$\{u,w,u_v\}=\sum_{m=0}^{\infty}\{u_m,w_m,u_{vm}\}\cos m\theta,\qquad \{v,u_h\}=\sum_{m=0}^{\infty}\{v_m,u_{hm}\}\sin m\theta, \tag{7a}$$

$$\left.\begin{aligned}
&\{\sigma_r,\sigma_\theta,\sigma_z,\tau_{rz},\tau_{vz}\}=\sum_{m=0}^{\infty}\{\sigma_{rm},\sigma_{\theta m},\sigma_{zm},\tau_{rzm},\tau_{vzm}\}\cos m\theta,\\
&\{\tau_{r\theta},\tau_{z\theta},\tau_{hz}\}=\sum_{m=0}^{\infty}\{\tau_{r\theta m},\tau_{z\theta m},\tau_{hzm}\}\sin m\theta,
\end{aligned}\right\} \tag{7b}$$

$$\{p,v_z\}=\sum_{m=0}^{\infty}\{p_m,v_{zm}\}\cos m\theta, \tag{7c}$$

$$\{q_z,q_{rz},q_{vz}\}=\sum_{m=0}^{\infty}\{q_{zm},q_{rzm},q_{vzm}\}\cos m\theta,\qquad \{q_{\theta z},q_{hz}\}=\sum_{m=0}^{\infty}\{q_{\theta zm},q_{hzm}\}\sin m\theta, \tag{7d}$$

where $q_{vz}=\frac{1}{r}\left[\frac{\partial(rq_{rz})}{\partial r}+\frac{\partial q_{\theta z}}{\partial\theta}\right]$, $q_{hz}=-\frac{1}{r}\left[\frac{\partial(rq_{\theta z})}{\partial r}-\frac{\partial q_{rz}}{\partial\theta}\right]$.

By substituting Eqs. (7a)–(7c) into Eqs. (6a) and (6b), we get the following two matrix partial differential equations:

$$\frac{\partial}{\partial z}\widetilde{\mathbf{X}}_m(r,z,s)=\tilde{\mathbf{A}}_m(r,s)\widetilde{\mathbf{X}}_m(r,z,s), \tag{8a}$$

$$\frac{\partial}{\partial z}\widetilde{\mathbf{Y}}_m(r,z,s)=\tilde{\mathbf{B}}_m(r,s)\widetilde{\mathbf{Y}}_m(r,z,s), \tag{8b}$$

where $\widetilde{\mathbf{X}}_m(r,z,s)=[\tilde{u}_{vm}(r,z,s),\tilde{\sigma}_{zm}(r,z,s),\tilde{p}_m(r,z,s),\tilde{w}_m(r,z,s),\tilde{\tau}_{vzm}(r,z,s),\tilde{v}_{zm}(r,z,s)]^{\mathrm{T}}$, and $\widetilde{\mathbf{Y}}_m(r,z,s)=[\tilde{u}_{hm}(r,z,s),\tilde{\tau}_{hzm}(r,z,s)]^{\mathrm{T}}$, and matrices $\tilde{\mathbf{A}}_m(r,s)$ and $\tilde{\mathbf{B}}_m(r,s)$ take the form

$$\tilde{\mathbf{A}}_m(r,s) = \begin{pmatrix} 0 & 0 & 0 & -\nabla_m^2 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{k_z} \\ -\frac{v}{1-v} & \frac{1-2v}{2\mu(1-v)} & \frac{1-2v}{2\mu(1-v)} & 0 & 0 & 0 \\ -\frac{2\mu}{1-v}\nabla_m^2 & -\frac{v}{1-v}\nabla_m^2 & \frac{1-2v}{1-v}\nabla_m^2 & 0 & 0 & 0 \\ -s\frac{1-2v}{1-v} & -s\frac{1-2v}{2\mu(1-v)} & \gamma^2 k_z \nabla_m^2 - s\left(\frac{1-2v}{2\mu(1-v)} + n\beta\right) & 0 & 0 & 0 \end{pmatrix}, \tag{9a}$$

$$\tilde{\mathbf{B}}_m(r,s) = \begin{pmatrix} 0 & \frac{1}{\mu} \\ -\mu\nabla_m^2 & 0 \end{pmatrix}, \tag{9b}$$

where $\nabla_m^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}$.

By performing Hankel transforms to Eqs. (9a) and (9b), we get two single-order ordinary differential matrix equations in the Laplace–Hankel transform domain

$$\frac{\mathrm{d}}{\mathrm{d}z}\overline{\mathbf{X}}_m(\xi,z,s) = \overline{\mathbf{A}}(\xi,s)\overline{\mathbf{X}}_m(\xi,z,s), \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\overline{\mathbf{Y}}_m(\xi,z,s) = \overline{\mathbf{B}}(\xi,s)\overline{\mathbf{Y}}_m(\xi,z,s), \tag{10b}$$

where $\overline{\mathbf{X}}_m(\xi,z,s) = [\bar{u}_{vm}(\xi,z,s), \bar{\sigma}_{zm}(\xi,z,s), \bar{p}_m(\xi,z,s), \bar{w}_m(\xi,z,s), \bar{\tau}_{vzm}(\xi,z,s), \bar{v}_{zm}(\xi,z,s)]^{\mathrm{T}}$, $\overline{\mathbf{Y}}_m(\xi,z,s) = [\bar{u}_{hm}(\xi,z,s), \bar{\tau}_{hzm}(\xi,z,s)]^{\mathrm{T}}$, and

$$\overline{\mathbf{X}}_m(\xi,z,s) = \int_0^\infty rJ_m(\xi r)\widetilde{\mathbf{X}}_m(r,z,s)\,\mathrm{d}r, \qquad \widetilde{\mathbf{X}}_m(r,z,s) = \int_0^\infty \xi J_m(\xi r)\overline{\mathbf{X}}_m(\xi,z,s)\,\mathrm{d}\xi, \tag{11a}$$

$$\overline{\mathbf{Y}}_m(\xi,z,s) = \int_0^\infty rJ_m(\xi r)\widetilde{Y}_m(r,z,s)\,\mathrm{d}r, \qquad \widetilde{\mathbf{Y}}_m(r,z,s) = \int_0^\infty \xi J_m(\xi r)\overline{\mathbf{Y}}_m(\xi,z,s)\,\mathrm{d}\xi \tag{11b}$$

where $J_m(\xi r)$ is the first kind of Bessel function of order m, and

$$\overline{\mathbf{A}}(\xi,s) = \begin{pmatrix} 0 & 0 & 0 & \xi^2 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{k_z} \\ -\frac{v}{1-v} & \frac{1-2v}{2\mu(1-v)} & \frac{1-2v}{2\mu(1-v)} & 0 & 0 & 0 \\ \frac{2\mu}{1-v}\xi^2 & \frac{v}{1-v}\xi^2 & -\frac{1-2v}{1-v}\xi^2 & 0 & 0 & 0 \\ -s\frac{1-2v}{1-v} & -s\frac{1-2v}{2\mu(1-v)} & -\gamma^2 k_z \xi^2 - s\left(\frac{1-2v}{2\mu(1-v)} + n\beta\right) & 0 & 0 & 0 \end{pmatrix}, \tag{12a}$$

$$\overline{\mathbf{B}}(\xi,s) = \begin{pmatrix} 0 & \frac{1}{\mu} \\ \mu\xi^2 & 0 \end{pmatrix}. \tag{12b}$$

Assuming there is no external load between depth z_1 and depth z_2, solutions of the ordinary differential matrix Eqs. (10a) and (10b) can be written as

$$\overline{\mathbf{X}}_m(\xi,z_2,s) = \mathbf{T}(\xi,z_2-z_1,s)\overline{\mathbf{X}}_m(\xi,z_1,s), \tag{13a}$$

$$\overline{\mathbf{Y}}_m(\xi,z_2,s) = \mathbf{S}(\xi,z_2-z_1,s)\overline{\mathbf{Y}}_m(\xi,z_1,s), \tag{13b}$$

where $\mathbf{T}(\xi,z,s)=\exp\left[z\overline{\mathbf{A}}(\xi,s)\right]$ and $\mathbf{S}(\xi,z,s)=\exp\left[z\overline{\mathbf{B}}(\xi,s)\right]$ are named transfer matrices because they transfer the solutions from depth z_1 to the depth z_2 (i.e. calculates the state vectors $\overline{\mathbf{X}}(\xi,z_2,s)$ and $\overline{\mathbf{Y}}(\xi,z_2,s)$ from the state vectors $\overline{\mathbf{X}}(\xi,z_1,s)$ and $\overline{\mathbf{Y}}(\xi,z_1,s)$).

The proper equations of the matrices $\overline{\mathbf{A}}(\xi,s)$ and $\overline{\mathbf{B}}(\xi,s)$ are

$$(\lambda_{\mathrm{A}}^2-\xi^2)^2\left\{\lambda_{\mathrm{A}}^2-\left(\gamma^2\xi^2+\frac{s(1-2v)}{2\mu(1-v)k_z}+\frac{sn\beta}{k_z}\right)\right\}=0, \tag{14a}$$

$$\lambda_{\mathrm{B}}^2-\xi^2=0, \tag{14b}$$

respectively, where $\lambda_{\mathrm{A}},\lambda_{\mathrm{B}}$ are eigenvalues of matrices $\overline{\mathbf{A}}(\xi,s)$ and $\overline{\mathbf{B}}(\xi,s)$, respectively. So matrix $\overline{\mathbf{A}}(\xi,s)$ has two equal eigenvalues ξ, two equal eigenvalues $-\xi$, and two eigenvalues $\pm\eta$, where

$$\eta=\sqrt{\gamma^2\xi^2+\frac{s(1-2v)}{2\mu(1-v)k_z}+\frac{sn\beta}{k_z}}, \tag{15}$$

and matrix $\overline{\mathbf{B}}(\xi,s)$ has two eigenvalues $\pm\xi$.

According to Cayley–Hamilton theorem, transfer matrices $\mathbf{T}$ and $\mathbf{S}$ can be expressed as

$$\mathbf{T}=\exp[z\overline{\mathbf{A}}(\xi,s)]=a_0\mathbf{E}_{6\times6}+a_1\overline{\mathbf{A}}+a_2\overline{\mathbf{A}}^2+a_3\overline{\mathbf{A}}^3+a_4\overline{\mathbf{A}}^4+a_5\overline{\mathbf{A}}^5, \tag{16a}$$

$$\mathbf{S}=\exp[z\overline{\mathbf{B}}(\xi,s)]=b_0\mathbf{E}_{2\times2}+b_1\overline{\mathbf{B}}. \tag{16b}$$

The equations which are obtained by substituting the eigenvalues λ_{A} or λ_{B} for the matrices $\overline{\mathbf{A}}(\xi,s)$ and $\overline{\mathbf{B}}(\xi,s)$ in Eqs. (16a) and (16b) should also be tenable, therefore we have

$$\left\{\begin{matrix} ch(\eta z)\\ sh(\eta z)\\ ch(\xi z)\\ sh(\xi z)\\ z\cdot ch(\xi z)\\ z\cdot sh(\xi z)\end{matrix}\right\}=\begin{bmatrix}1&0&\eta^2&0&\eta^4&0\\0&\eta&0&\eta^3&0&\eta^5\\1&0&\xi^2&0&\xi^4&0\\0&\xi&0&\xi^3&0&\xi^5\\0&1&0&3\xi^2&0&5\xi^4\\0&0&2\xi&0&4\xi^3&0\end{bmatrix}\left\{\begin{matrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_5\end{matrix}\right\}, \tag{17a}$$

$$b_0=ch\xi z,\qquad b_1=\frac{sh\xi z}{\xi}. \tag{17b}$$

Transfer matrices $\mathbf{T}$ and $\mathbf{S}$ are therefore obtained by substituting a_0, a_1, a_2, a_3, a_4, a_5 (solved from Eq. (17a)) into Eq. (16a), and (17b) into Eq. (16b). Expressions of the elements of $\mathbf{T}$ and $\mathbf{S}$ are included in Appendix A.

As an example, we apply the solutions in Eqs. (13a) and (13b) in the soil layer shown in Fig. 1:

(1) Domain 1 (which is bounded by $0\leqslant z\leqslant h$)
By utilizing the forward transfer matrix technology, we get

$$\overline{\mathbf{X}}_m(\xi,z,s)=\mathbf{T}(\xi,z,s)\overline{\mathbf{X}}_m(\xi,0,s),\qquad \overline{\mathbf{Y}}_m(\xi,z,s)=\mathbf{S}(\xi,z,s)\overline{\mathbf{Y}}_m(\xi,0,s). \tag{18}$$

(2) Domain 2 (which is bounded by $h<z\leqslant H$)
By utilizing the backward transfer matrix technology, we get

$$\overline{\mathbf{X}}_m(\xi,z,s)=\mathbf{T}(\xi,z-H,s)\overline{\mathbf{X}}_m(\xi,H,s),\qquad \overline{\mathbf{Y}}_m(\xi,z,s)=\mathbf{S}(\xi,z-H,s)\overline{\mathbf{Y}}_m(\xi,H,s). \tag{19}$$

The solution at each given point of the domain by using Eqs. (18) and (19) requires the knowledge of the state vectors of the transformed variables at the top of the layer, $\overline{\mathbf{X}}_m(\xi,0,s)$, $\overline{\mathbf{Y}}_m(\xi,0,s)$ and at the base of the layer, $\overline{\mathbf{X}}_m(\xi, H, s)$, $\overline{\mathbf{Y}}_m(\xi, H, s)$. For the case in Fig. 1, six variables should be known as boundary conditions, the remaining six variables must be calculated by the following two relationships that exist between the state vectors $\overline{\mathbf{X}}(\xi, 0, s)$, $\overline{\mathbf{Y}}(\xi, 0, s)$ and $\overline{\mathbf{X}}(\xi, H, s)$, $\overline{\mathbf{Y}}(\xi, H, s)$

$$\overline{\mathbf{X}}_m(\xi, H, s) = \mathbf{T}(\xi, H, s)\overline{\mathbf{X}}_m(\xi, 0, s) + \mathbf{T}(\xi, H - h, s)\overline{\mathbf{Q}}_{Xm}(\xi, s), \tag{20a}$$

$$\overline{\mathbf{Y}}_m(\xi, H, s) = \mathbf{S}(\xi, H, s)\overline{\mathbf{Y}}_m(\xi, 0, s) + \mathbf{S}(\xi, H - h, s)\overline{\mathbf{Q}}_{Ym}(\xi, s), \tag{20b}$$

where $\overline{\mathbf{Q}}_{Xm}(\xi, s) = \{0, \bar{q}_{zm}(\xi, s), 0, 0, \bar{q}_{vzm}(\xi, s), 0\}^{\mathrm{T}}$ and $\overline{\mathbf{Q}}_{Ym}(\xi, s) = \{0, \bar{q}_{hzm}(\xi, s)\}^{\mathrm{T}}$.

Note that, in the following, we will simplify $\overline{\mathbf{X}}_m(\xi, z, s)$, $\mathbf{T}(\xi, z, s)$, $\overline{\mathbf{Y}}_m(\xi, z, s)$, $\mathbf{S}(\xi, z, s)$, etc. into $\overline{\mathbf{X}}_m(z)$, $\mathbf{T}(z)$, $\overline{\mathbf{Y}}_m(z)$, $\mathbf{S}(z)$, etc.

3. Solutions of multilayered poroelastic half space

The problem considered here is a multilayered poroelastic half space with high saturation degree and loaded by q_z, q_{rz}, $q_{\theta z}$ at depth h_q below the surface (see Fig. 2). The multilayered half space consists of $n+1$ perfectly bonded poroelastic layers which are infinite in horizontal extent. Each layer is homogeneous, the load is located in the ith soil layer occupying the region $Z_{i-1} \leqslant z \leqslant Z_i$ with thickness $H_i = Z_i - Z_{i-1}$. For layer $j = 1, 2, \ldots, n$, it has the thickness H_j, the shear modulus μ_j, Poisson's ratio v_j, compressibility of pore fluid β_j and the permeability parameters k_{rj}, k_{zj}, γ_j^2. For layer $n+1$, it occupies the region $Z_n \leqslant z < \infty$, and has the poroelastic parameters μ_{n+1}, v_{n+1}, β_{n+1} and the permeability parameters $k_{r,n+1}$, $k_{z,n+1}$, γ_{n+1}^2.

3.1. Boundary conditions

3.1.1. z = 0

The surface $z = 0$ of the multilayered half space is considered as traction free, which takes the following mechanical boundary conditions in the transform domain

$$\bar{\sigma}_{zm}(\xi, 0, s) = \bar{\tau}_{vzm}(\xi, 0, s) = \bar{\tau}_{hzm}(\xi, 0, s) = 0, \tag{21a}$$

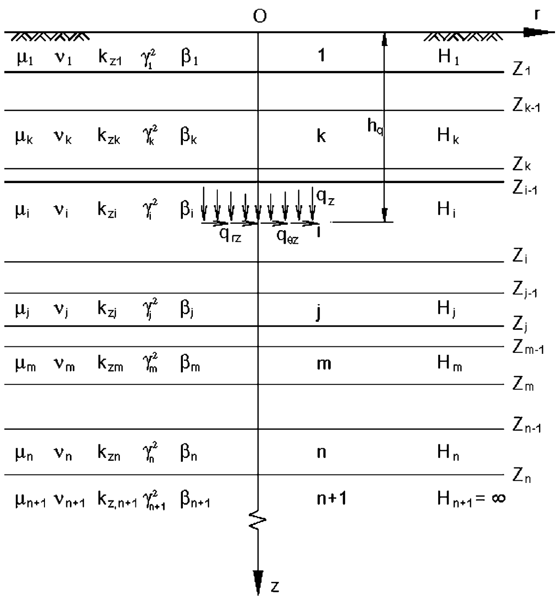

Fig. 2. A multilayered poroelastic half space subjected to external load.

for a permeable surface, it has the following hydraulic boundary condition

$$\bar{p}_m(\xi, 0, s) = 0, \tag{21b}$$

while for an impermeable surface, it has the following hydraulic boundary condition

$$\bar{v}_{zm}(\xi, 0, s) = 0. \tag{21c}$$

3.1.2. $z = Z_n$

The half space extends to infinity, in which the displacement and water flow velocity fields should reduce to zero and the stresses and pore pressure should be bounded as $z \to +\infty$. Then in order to satisfy such natural regularity conditions, we have the following equations for vectors $\overline{\mathbf{X}}(Z_n)$, $\overline{\mathbf{Y}}(Z_n)$:

$$\boldsymbol{\varphi}_{n+1}\overline{\mathbf{X}}(Z_n) = \mathbf{0}, \qquad \boldsymbol{\psi}_{n+1}\overline{\mathbf{Y}}(Z_n) = \mathbf{0}, \tag{22}$$

where the matrices $\boldsymbol{\varphi}_{n+1}$ (dimensions 3×6) and $\boldsymbol{\psi}_{n+1}$ (dimensions 1×2) are derived and defined in Appendix B.

For the limiting case of $\mu_{n+1} \to \infty$, substituting $\mu_{n+1} \to \infty$ into (22) yields

$$\bar{u}_{vm}(\xi, Z_n, s) = \bar{w}_m(\xi, Z_n, s) = \bar{u}_{hm}(\xi, Z_n, s) = 0, \tag{23}$$

then the infinite layer $n+1$ is simplified into a rigid and rough mechanical boundary condition.

For the limiting case of $k_{z,n+1} \to 0$, substituting $k_{z,n+1} \to 0$ into (22) yields

$$\bar{v}_{zm}(\xi, Z_n, s) = 0, \tag{24a}$$

then the infinite layer $n+1$ is simplified into an impermeable hydraulic boundary condition.

For the limiting case of $k_{z,n+1} \to \infty$, substituting $k_{z,n+1} \to \infty$ into (22) yields

$$\bar{p}_m(\xi, Z_n, s) = 0, \tag{24b}$$

then the infinite layer $n+1$ is simplified into a permeable hydraulic boundary condition.

3.2. Continuity condition

The layers are perfectly bonded with the following interfacial continuity conditions for any interface between the jth layer and $(j+1)$th layer

$$\overline{\mathbf{X}}_m(Z_j^-) = \overline{\mathbf{X}}_m(Z_j^+), \qquad \overline{\mathbf{Y}}_m(Z_j^-) = \overline{\mathbf{Y}}_m(Z_j^+). \tag{25}$$

3.3. Solutions of boundary vectors $\overline{\mathbf{X}}_m(0)$, $\overline{\mathbf{X}}_m(Z_n)$ and $\overline{\mathbf{Y}}_m(0)$, $\overline{\mathbf{Y}}_m(Z_n)$

By using Eqs. (18) and (19) and the continuity conditions (25) between any two connected layers we can express the vectors $\overline{\mathbf{X}}_m(Z_{i-1})$, $\overline{\mathbf{X}}_m(Z_i)$, $\overline{\mathbf{X}}_m(Z_{i-1})$, $\overline{\mathbf{Y}}_m(Z_i)$ by the vectors at $z = 0$ (i.e. $\overline{\mathbf{X}}_m(0)$, $\overline{\mathbf{Y}}_m(0)$) and $z = Z_n$ (i.e. $\overline{\mathbf{X}}_m(Z_n)$, $\overline{\mathbf{Y}}_m(Z_n)$) as

$$\overline{\mathbf{X}}_m(Z_{i-1}) = \mathbf{T}_{i-1}(H_{i-1})\cdots\mathbf{T}_2(H_2)\mathbf{T}_1(H_1)\overline{\mathbf{X}}_m(0), \tag{26a}$$

$$\overline{\mathbf{X}}_m(Z_i) = \mathbf{T}_{i+1}(-H_{i+1})\mathbf{T}_{i+2}(-H_{i+2})\cdots\mathbf{T}_{n-1}(-H_{n-1})\mathbf{T}_n(-H_n)\overline{\mathbf{X}}_m(Z_n), \tag{26b}$$

$$\overline{\mathbf{Y}}_m(Z_{i-1}) = \mathbf{S}_{i-1}(H_{i-1})\cdots\mathbf{S}_2(H_2)\mathbf{S}_1(H_1)\overline{\mathbf{Y}}_m(0), \tag{27a}$$

$$\overline{\mathbf{Y}}_m(Z_i) = \mathbf{S}_{i+1}(-H_{i+1})\mathbf{S}_{i+2}(-H_{i+2})\cdots\mathbf{S}_{n-1}(-H_{n-1})\mathbf{S}_n(-H_n)\overline{\mathbf{Y}}_m(Z_n), \tag{27b}$$

and the four vectors $\overline{\mathbf{X}}_m(Z_{i-1})$, $\overline{\mathbf{X}}_m(Z_i)$, $\overline{\mathbf{Y}}_m(Z_{i-1})$, $\overline{\mathbf{Y}}_m(Z_i)$ should satisfy the following equations according to Eqs. (20a) and (20b):

$$\overline{\mathbf{X}}_m(Z_i) = \mathbf{T}_i(H_i)\overline{\mathbf{X}}_m(Z_{i-1}) + \mathbf{T}_i(Z_i - h_q)\overline{\mathbf{Q}}_{Xm}(\xi, s), \tag{28a}$$

$$\overline{\mathbf{Y}}_m(Z_i) = \mathbf{S}_i(H_i)\overline{\mathbf{Y}}_m(Z_{i-1}) + \mathbf{S}_i(Z_i - h_q)\overline{\mathbf{Q}}_{Ym}(\xi, s), \tag{28b}$$

where $\overline{\mathbf{Q}}_{Xm}(\xi, s) = \{0, \bar{q}_{zm}(\xi, s), 0, 0, \bar{q}_{vzm}(\xi, s), 0\}^{\mathrm{T}}$ and $\overline{\mathbf{Q}}_{Ym}(\xi, s) = \{0, \bar{q}_{hzm}(\xi, s)\}^{\mathrm{T}}$.

Substituting Eqs. (26a), (26b) and (27a), (27b) into Eqs. (28a) and (28b), respectively, we get

$$\begin{aligned}&\mathbf{T}_{i+1}(-H_{i+1})\mathbf{T}_{i+2}(-H_{i+2})\cdots\mathbf{T}_{n-1}(-H_{n-1})\mathbf{T}_n(-H_n)\overline{\mathbf{X}}_m(Z_n)\\&\quad = \mathbf{T}_i(H_i)\mathbf{T}_{i-1}(H_{i-1})\cdots\mathbf{T}_2(H_2)\mathbf{T}_1(H_1)\overline{\mathbf{X}}_m(0) + \mathbf{T}_i(Z_i - h_q)\overline{\mathbf{Q}}_{Xm}(\xi, s),\end{aligned} \tag{29a}$$

$$\begin{aligned}&\mathbf{S}_{i+1}(-H_{i+1})\mathbf{S}_{i+2}(-H_{i+2})\cdots\mathbf{S}_{n-1}(-H_{n-1})\mathbf{S}_n(-H_n)\overline{\mathbf{Y}}_m(Z_n)\\&\quad = \mathbf{S}_i(H_i)\mathbf{S}_{i-1}(H_{i-1})\cdots\mathbf{S}_2(H_2)\mathbf{S}_1(H_1)\overline{\mathbf{Y}}_m(0) + \mathbf{S}_i(Z_i - h_q)\overline{\mathbf{Q}}_{Ym}(\xi, s).\end{aligned} \tag{29b}$$

By using the matrix behavior $\mathbf{T}(-z) = [\mathbf{T}(z)]^{-1}$ and $\mathbf{S}(-z) = [\mathbf{S}(z)]^{-1}$ we can change Eqs. (29a) and (29b) into

$$\overline{\mathbf{X}}_m(Z_n) = \mathbf{M}_x \cdot \overline{\mathbf{X}}_m(0) + \mathbf{N}_x, \qquad \overline{\mathbf{Y}}_m(Z_n) = \mathbf{M}_y \cdot \overline{\mathbf{Y}}_m(0) + \mathbf{N}_y, \tag{30}$$

where the matrices $\mathbf{M}_x = \mathbf{T}_n(H_n)\cdots\mathbf{T}_i(H_i)\cdots\mathbf{T}_1(H_1)$, $\mathbf{N}_x = \mathbf{T}_n(H_n)\cdots\mathbf{T}_{i+1}(H_{i+1})\mathbf{T}_i(Z_i - h_q)\overline{\mathbf{Q}}_{Xm}(\xi, s)$, and $\mathbf{M}_y = \mathbf{S}_n(H_n)\cdots\mathbf{S}_i(H_i)\cdots S_1(H_1)$, $\mathbf{N}_y = \mathbf{S}_n(H_n)\cdots\mathbf{S}_{i+1}(H_{i+1})S_i(Z_i - h_q)\overline{\mathbf{Q}}_{Ym}(\xi, s)$.

According to Eq. (30) and the regularity conditions in Eq. (22), we have

$$\boldsymbol{\varphi}_{n+1}\mathbf{M}_x \cdot \overline{\mathbf{X}}_m(0) + \boldsymbol{\varphi}_{n+1}\mathbf{N}_x = 0, \qquad \boldsymbol{\psi}_{n+1}\mathbf{M}_y \cdot \overline{\mathbf{Y}}_m(0) + \boldsymbol{\psi}_{n+1}\mathbf{N}_y = 0. \tag{31}$$

By utilizing the four known boundary variables at the top of the media included in Eqs. (21a)–(21c), we can solve the other four unknown boundary variables in vectors $\overline{\mathbf{X}}_m(0)$, $\overline{\mathbf{Y}}_m(0)$ from Eq. (31). Then from Eq. (30), we can obtain vectors $\overline{\mathbf{X}}_m(Z_n)$, $\overline{\mathbf{Y}}_m(Z_n)$.

3.4. Solutions of vectors $\overline{\mathbf{X}}_m(z)$, $\overline{\mathbf{Y}}_m(z)$ in the domain which is bounded by $0 \leqslant z \leqslant h_q$

For $Z_{k-1} \leqslant z < Z_k$ and $0 \leqslant z \leqslant h_q$, we can express the vectors $\overline{\mathbf{X}}_m(z)$ and $\overline{\mathbf{Y}}_m(z)$ in the multilayered half space in terms of the vectors at $z = 0$ (i.e. $\overline{\mathbf{X}}_m(0)$ and $\overline{\mathbf{Y}}_m(0)$) via the forward transfer matrix technique. By using the continuity conditions (25), we obtain

$$\overline{\mathbf{X}}_m(z) = \mathbf{T}_{z1}(z)\overline{\mathbf{X}}_m(0), \qquad \overline{\mathbf{Y}}_m(z) = \mathbf{S}_{z1}(z)\overline{\mathbf{Y}}_m(0), \tag{32}$$

where $\mathbf{T}_{z1}(z) = \mathbf{T}_k(z - Z_{k-1})\mathbf{T}_{k-1}(H_{k-1})\cdots\mathbf{T}_1(H_1)$ and $\mathbf{S}_{z1}(z) = \mathbf{S}_k(z - Z_{k-1})\mathbf{S}_{k-1}(H_{k-1})\cdots\mathbf{S}_1(H_1)$.

3.5. Solutions of vectors $\overline{\mathbf{X}}_m(z)$ and $\overline{\mathbf{Y}}_m(z)$ in the domain which is bounded by $h_q < z \leqslant Z_n$

For $Z_{m-1} \leqslant z < Z_m$ and $h_q < z \leqslant Z_n$, we can express the vectors $\overline{\mathbf{X}}_m(z)$ and $\overline{\mathbf{Y}}_m(z)$ in the multilayered half space in terms of the vector at $z = Z_n$ (i.e. $\overline{\mathbf{X}}_m(Z_n)$ and $\overline{\mathbf{Y}}_m(Z_n)$) via the backward transfer matrix technique. By using the continuity conditions (25), we get

$$\overline{\mathbf{X}}_m(z) = \mathbf{T}_{z2}(z)\overline{\mathbf{X}}_m(Z_n), \qquad \overline{\mathbf{Y}}_m(z) = \mathbf{S}_{z2}(z)\overline{\mathbf{Y}}_m(Z_n), \tag{33}$$

where $\mathbf{T}_{z2}(z) = \mathbf{T}_m(z - Z_m)\mathbf{T}_{m+1}(-H_{m+1})\cdots\mathbf{T}_n(-H_n)$ and $\mathbf{S}_{z2}(z) = \mathbf{S}_m(z - Z_m)\mathbf{S}_{m+1}(-H_{m+1})\cdots S_n(-H_n)$.

3.6. Solutions of vectors $\overline{\mathbf{X}}_m(z)$ and $\overline{\mathbf{Y}}_m(z)$ in the domain which is bounded by $Z_n < z < +\infty$

For $Z_n < z < +\infty$, to guarantee the boundedness of the solution, the function of exponential growth must vanish in the transfer matrix. Therefore, the vectors $\overline{\mathbf{X}}_m(z)$ and $\overline{\mathbf{Y}}_m(z)$ can be expressed in terms of the vectors at $z = Z_n$ (i.e. $\overline{\mathbf{X}}_m(Z_n)$ and $\overline{\mathbf{Y}}_m(Z_n)$) as

$$\overline{\mathbf{X}}_m(z) = \mathbf{T}_{z3}(z)\overline{\mathbf{X}}_m(Z_n), \qquad \overline{\mathbf{Y}}_m(z) = \mathbf{S}_{z3}(z)\overline{\mathbf{Y}}_m(Z_n), \tag{34}$$

where matrices $\mathbf{T}_{z3}(z) = \mathbf{T}'_{n+1}(z - Z_n)$ and $\mathbf{S}_{z3}(z) = \mathbf{S}'_{n+1}(z - Z_n)$. Transfer matrices $\mathbf{T}'_{n+1}(z - Z_n)$ and $\mathbf{S}'_{n+1}(z - Z_n)$ are calculated according to Appendix A with $ch\xi z$, $sh\xi z$ and $ch\eta z$, $sh\eta z$ replaced by $e^{-\xi z}/2$, $-e^{-\xi z}/2$ and $e^{-\eta z}/2$, $-e^{-\eta z}/2$, respectively.

3.7. Solutions of vectors $\mathbf{X}_m(r,z,t)$ and $\mathbf{Y}_m(r,z,t)$ in the physical domain

By performing inverse Hankel–Laplace transforms to Eqs. (32)–(34), we can obtain the solutions of $\mathbf{X}_m(r,z,t)$ and $\mathbf{Y}_m(r,z,t)$ in the physical domain.

A numerical method for evaluating inverse Hankel transform is presented. The function $J_m(\xi r)$ is wave function, and converges very slowly, therefore the semi-infinite integral of the inverse Hankel transform is discretized into a set of intervals according to the zero points of the function $J_m(\xi r)$

$$\widetilde{\mathbf{X}}_m(r,z,s) = \int_0^{\xi_1} \xi J_m(\xi r)\overline{\mathbf{X}}_m(\xi,z,s)\,\mathrm{d}\xi + \sum_{m=1}^{\infty} \int_{\xi_m}^{\xi_{m+1}} \xi J_m(\xi r)\overline{\mathbf{X}}_m(\xi,z,s)\,\mathrm{d}\xi, \tag{35a}$$

$$\widetilde{\mathbf{Y}}_m(r,z,s) = \int_0^{\xi_1} \xi J_m(\xi r)\overline{\mathbf{Y}}_m(\xi,z,s)\,\mathrm{d}\xi + \sum_{m=1}^{\infty} \int_{\xi_m}^{\xi_{m+1}} \xi J_m(\xi r)\overline{\mathbf{Y}}_m(\xi,z,s)\,\mathrm{d}\xi, \tag{35b}$$

where $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are the zero points of the function $J_m(\xi r)$, each term on the right side of Eqs. (35a) and (35b) can be integrated based on an adaptively iterative Simpson's (i.e. 3-point Gauss integration) quadrature technique. Enough accuracy for the semi-infinite integral can be obtained by taking the initial seven to eight terms in Eqs. (35a) and (35b).

By using a numerical scheme of Crump's method, solutions in the time domain can be obtained by inverting the solutions in the Laplace transform domain such as equations in (35a) and (35b) with high efficiency and accuracy.

4. Parametric study and numerical results

The numerical results of primary interest to geotechnical applications relate to the evaluation of the soil surface subsidence and excess pore pressure induced by the external load. In this section, example 1 compares the results obtained from the analysis of the same problem by using the solution in this paper and the solution presented in paper by Booker and Small (1987). Example 2 is presented to investigate the influences of anisotropy of permeability and compressibility of pore fluid on the subsidence at point with $r = z = 0$ and excess pore pressure along the line with $r = 0$ of a homogeneous single layer. Example 3 is given to show the efficiency of the present study to calculate a multilayered half space.

4.1. Example 1: Validation of the solutions

A consolidation problem is studied to compare the efficiency and accuracy of the proposed procedure against other existing results. A two-layered soil loaded by circular and uniform load as shown in Fig. 3 is

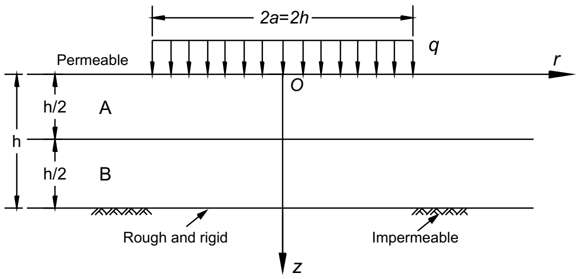

Fig. 3. A two-layered system subjected to circular and uniform load.

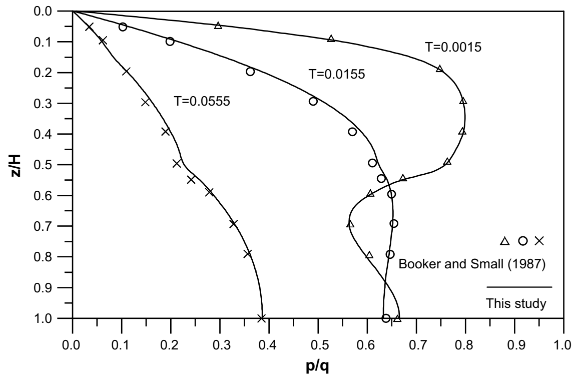

Fig. 4. Excess pore water pressure along the central line.

examined, the permeability of the two layers is isotropic, and the pore fluid is incompressible. The ratio of the permeability between layer A and layer B is $k_A/k_B = 4$ and the ratio of the shear modulus between layer A and layer B is $\mu_A/\mu_B = 1/4$, Poisson's ratio was chosen to be $v = 0.3$ for both layers.

Excess pore pressure along the central line with $r = 0$ at three different time factors $T = k_A\mu_A t/a^2 = 0.0015$, 0.0155, 0.0555 is presented in Fig. 4. Good agreement can be observed between results by this study and results by Booker and Small (1987).

4.2. Example 2: Parametric study for a single soil layer

A homogeneous single layer with free drainage surface overlying a rigid, rough and impermeable base is here studied, the soil layer is loaded by circular and uniform load q with radius a, and the thickness of the soil layer is $H = a$. The influence of the anisotropy of the permeability and the compressibility of the pore fluid is investigated.

(1) *Effects of hydraulic anisotropy*. Hydraulic anisotropy, $\gamma^2 = k_r/k_z$, describes the ratio of the horizontal permeability coefficient to the vertical permeability coefficient. For the fixed value of $\beta = \beta_0 = 4.5 \times 10^{-4}$ MPa^{-1} (this value corresponds to true bulk modulus of elasticity of water) and $v = 0.25$, four values of $\gamma^2 = (0.01, 1, 10, 100)$ are selected to study the influence of γ^2 on the consolidation.

The calculated dimensionless subsidence $\mu w(0, 0, T)/qa$ versus the dimensionless time factor $T = k_z\mu t/a^2$ is illustrated in Fig. 5, and the calculated dimensionless pore pressure p/q versus the dimensionless depth z/H at three different time factors $T = 0.001$, 0.01, 0.1 is illustrated in Fig. 6(a)–(c). Consolidation process is observed to be faster with the increase of γ^2, this is because soil with bigger γ^2 is more permeable, and excess

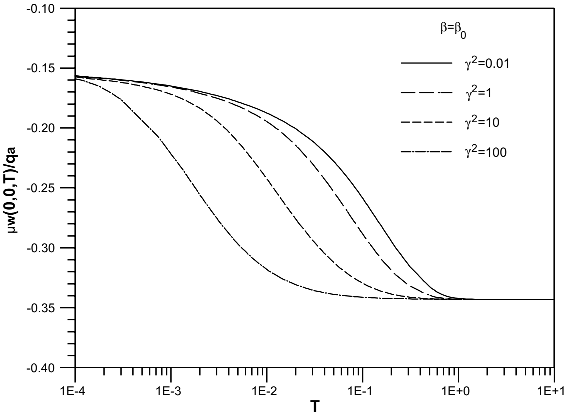

Fig. 5. Influence of permeability anisotropy parameter γ^2 on time-settlement behavior.

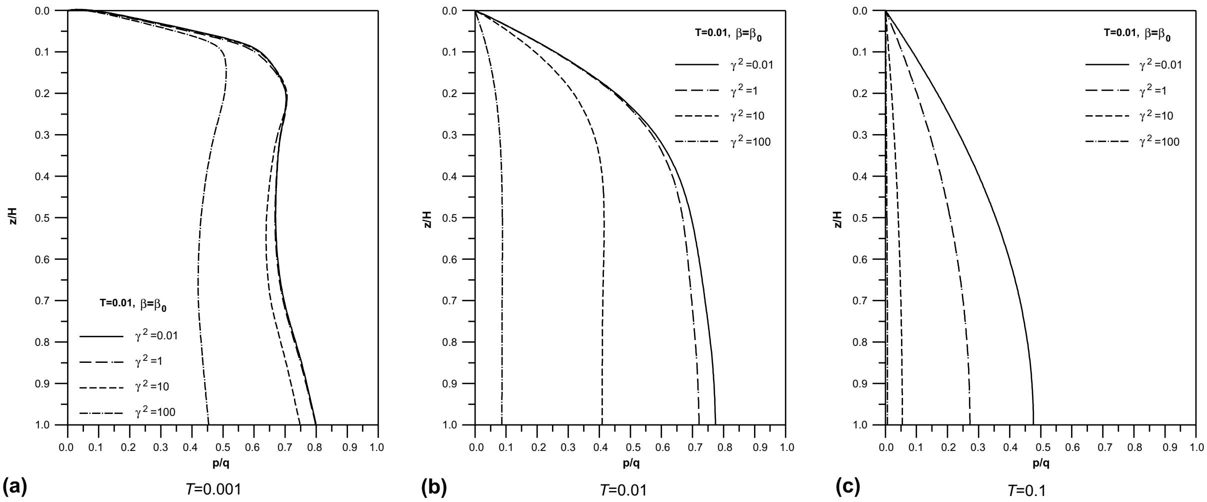

Fig. 6. Influence of permeability anisotropy parameter γ^2 on excess pore water pressure along the central line.

pore pressure dissipates faster. Besides it is found from Fig. 5 that the consolidation with different γ^2 has the same initial and final settlement.

(2) *Effects of compressibility of pore fluid.* For the fixed values of $\gamma^2 = 1$ and $v = 0.25$, four values of $\beta = (1, 100, 300, 1000)\beta_0$ (which corresponds to saturation degree $S_r = 100\%$, 99.55%, 98.65%, 95.5%, see Eq. (1c)) are selected to study the influence of compressibility of pore fluid on the consolidation, K_w is taken as 2.22×10^3 MPa, P_{w0} is absolute water pressure (taken to be 0.1 MPa).

The calculated dimensionless subsidence $\mu w(0,0,T)/qa$ versus the dimensionless time factor $T = k_z \mu t/a^2$ is illustrated in Fig. 7, and the calculated dimensionless pore pressure p/q versus the dimensionless depth z/H at three different time factors $T = 0.001$, 0.01, 0.1 are illustrated in Fig. 8(a)–(c). From Fig. 7 it can be seen that with the increase of the compressibility of the pore fluid, the final settlement is the same while the initial settlement increases. Fig. 8(a)–(c) show that the pore pressure dissipates faster with the increases of pore fluid compressibility.

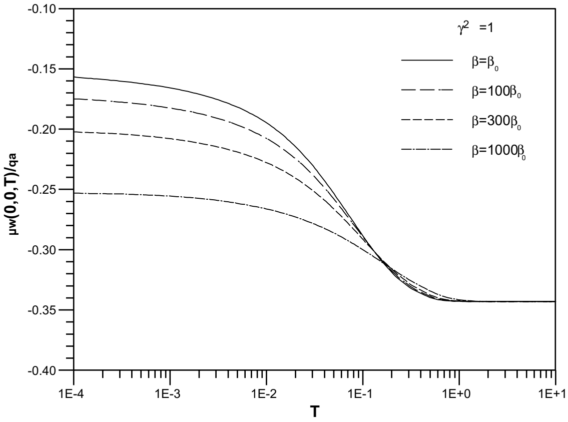

Fig. 7. Influence of pore fluid compressibility β on time-settlement behavior.

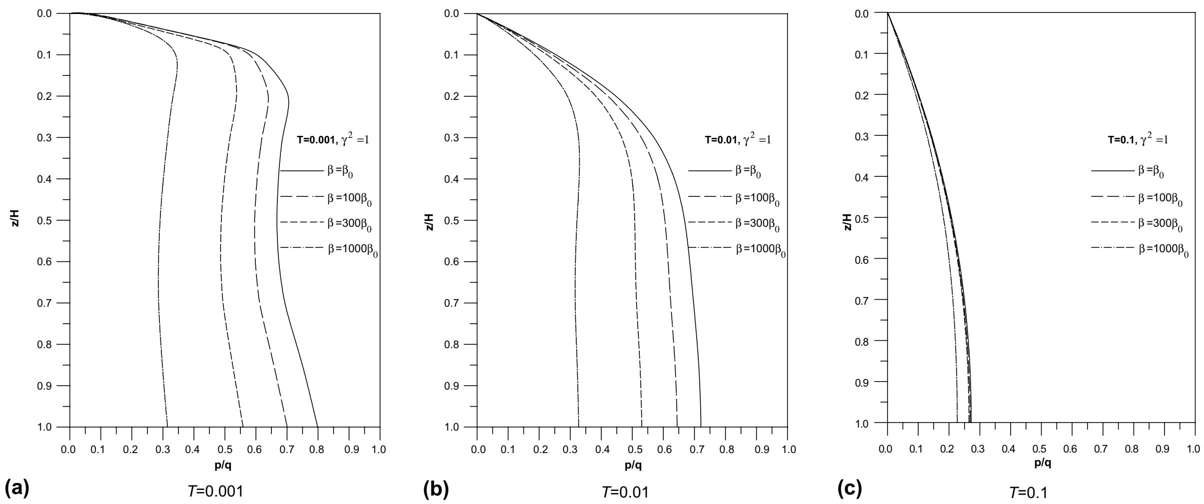

Fig. 8. Influence of pore fluid compressibility β on excess pore water pressure along the central line.

The above consolidation behavior related to compressibility of pore fluid can be explained as: after loading, the initial excess pore pressure generated in soil layer is the same, so for soil with bigger pore fluid compressibility, its volume becomes smaller which results in bigger initial settlement.

4.3. Example 3: Analysis of a multilayered half space

In order to illustrate the efficiency of the present analytical method to calculate the multilayered soil, a five-layered soil with free drainage surface overlying a homogeneous half space is here investigated, and the external uniform and circular load with diameter $5a$ is applied at soil surface. Soil depth, shear modulus, permeability coefficient, Poisson's ratio, degree of permeability coefficient, compressibility of pore fluid are shown in Fig. 9.

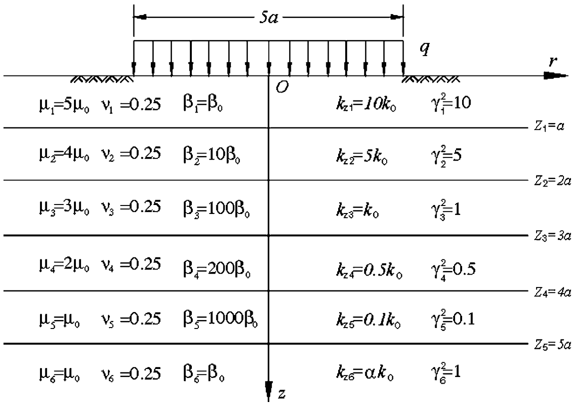

Fig. 9. A five-layered soil overlying half space subjected to circular and uniform load.

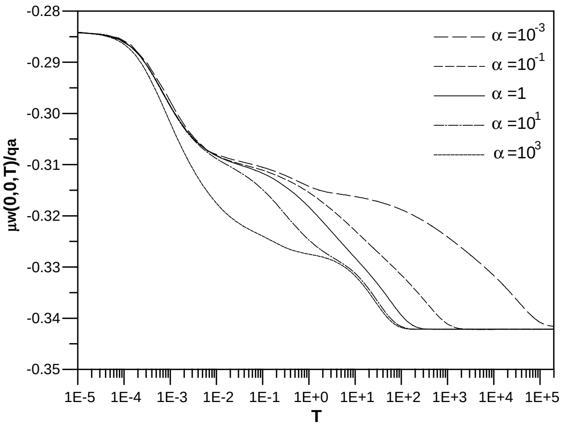

Fig. 10. Time-settlement behavior of a five-layered soil overlying a half space with five different values of α.

The dimensionless subsidence of point O is defined as $\mu_0 w(0,0,T)/qa$, and dimensionless time factor is defined as $T = k_0\mu_0 t/a^2$. Selected results are presented in Fig. 10, and different permeability coefficient ratios $\alpha = 10^{-3}, 10^{-1}, 1, 10^1, 10^3$ of the half space are selected. It is observed that the permeability of the half space has great influence on the settlement history, and with the increase of permeability coefficient of the half space consolidation becomes faster.

The above calculation at any point takes less than three seconds of CPU time for a Pentium III 1000 MHz PC.

5. Conclusions

In this paper, an analytical solution for multilayered poroelastic half space subjected to external load has been presented by utilizing state vector method, Laplace–Hankel integral transforms techniques and transfer matrix method. The permeability anisotropy and pore fluid compressibility are considered in the solution. The multilayered media with finite thickness is a limit case of this study.

The correctness of the present study is confirmed from the analysis of the same problem by using the results calculated by this method and other available results.

For a single soil layer, a group of numerical results are provided to examine the roles of permeability anisotropy, pore fluid compressibility on the consolidation process. These numerical results show that (i) The anisotropy of permeability does not have influence on the initial and final settlement, but it has much influence on the consolidation process; (ii) with the increase of the pore fluid compressibility, the soil will have bigger initial settlement, but it does not have effect on the final settlement. It may therefore be concluded that anisotropy of the permeability, compressibility of pore fluid must be properly considered if reasonable prediction of the consolidation process is to be obtained.

The analysis of a multilayered soil shows the high efficiency of the present method, and shows that the hydraulic behavior of the underlying half space has significant influence on the consolidation process.

Acknowledgements

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Appendix A. Elements of transfer matrices T and S

A.1. Elements of transfer matrix T

$$T_{11} = 2\mu\xi T_{12} + ch\xi z;$$

$$2\mu\xi T_{12} = \frac{\delta^2}{2\mu k_z} s\phi_0^2[ch\xi z - ch\eta z] + \phi_1 \xi z sh\xi z;$$

$$T_{13} = \frac{\delta}{2\mu}\phi_0[ch\xi z - ch\eta z];$$

$$T_{14} = \frac{\delta}{2}\phi_2 sh\xi z - \frac{\delta^2}{2\mu k_z}\frac{s\xi}{\eta}\phi_0^2 sh\eta z + \phi_1 \xi z ch\xi z;$$

$$2\mu\xi T_{15} = T_{14} + sh\xi z;$$

$$T_{16} = -\frac{\delta}{2\mu k_z}\frac{\phi_0}{\xi}\left[sh\xi z - \frac{\xi}{\eta} sh\eta z\right];$$

$$\frac{T_{21}}{2\mu\xi} = -2\mu\xi T_{12};$$

$$T_{22} = \frac{T_{21}}{2\mu\xi} + ch\xi z;$$

$$\frac{T_{23}}{2\mu\xi} = -T_{13};$$

$$\frac{T_{24}}{2\mu\xi} = T_{25} + sh\xi z;$$

$$T_{25} = -T_{14};$$

$$\frac{T_{26}}{2\mu\xi} = -T_{16};$$

$$T_{31} = -\frac{\delta}{k_z}s\phi_0[ch\xi z - ch\eta z];$$

$$2\mu\xi T_{32} = T_{31};$$

$$T_{33} = ch\eta z;$$

$$T_{34} = -\frac{\delta}{k_z}s\phi_0\left[sh\xi z - \frac{\xi}{\eta}sh\eta z\right];$$

$$2\mu\xi T_{35} = T_{34};$$

$$T_{36} = -\frac{1}{k_z\eta}sh\eta z;$$

$$T_{41} = \frac{\delta}{2}\phi_3 sh\xi z + \frac{\delta^2}{2\mu k_z}\frac{s\eta}{\xi}\phi_0^2 sh\eta z - \phi_1\xi zch\xi z;$$

$$2\mu\xi T_{42} = T_{41} + sh\xi z;$$

$$T_{43} = -\frac{\delta}{2\mu}\phi_0\left[sh\xi z - \frac{\eta}{\xi}sh\eta z\right];$$

$$T_{44} = 2\mu\xi T_{45} + ch\xi z;$$

$$T_{45} = -T_{12};$$

$$T_{46} = \frac{\delta}{2\mu k_z}\frac{\phi_0}{\xi}[ch\xi z - ch\eta z];$$

$$\frac{T_{51}}{2\mu\xi} = T_{52} + sh\xi z;$$

$$T_{52} = -T_{41};$$

$$\frac{T_{53}}{2\mu\xi} = -T_{43};$$

$$\frac{T_{54}}{2\mu\xi} = -2\mu\xi T_{45};$$

$$T_{55} = \frac{T_{54}}{2\mu\xi} + ch\xi z;$$

$$\frac{T_{56}}{2\mu\xi} = -T_{46};$$

$$T_{61} = \delta s\phi_0\xi\left[sh\xi z - \frac{\eta}{\xi}sh\eta z\right];$$

$$2\mu\xi T_{62} = T_{61};$$

$$T_{63} = -k_z\eta sh\eta z;$$

$$T_{64} = \delta s\phi_0\xi[ch\xi z - ch\eta z];$$

$$2\mu\xi T_{65} = T_{64};$$

$$T_{66} = ch\eta z.$$

where $\delta = \frac{1-2v}{1-v}$, $\phi_0(\xi,s) = \frac{\xi}{\eta^2-\xi^2}$, $\phi_1(\xi,s) = \frac{1}{2(1-v)}\left[1 + \frac{(1-2v)^2}{2\mu(1-v)k_z}\frac{s}{\eta^2-\xi^2}\right]$, $\phi_2(\xi,s) = 1 - \frac{1-2v}{2\mu(1-v)k_z}\frac{s\left(\eta^2-3\xi^2\right)}{\left(\eta^2-\xi^2\right)^2}$ and $\phi_3(\xi,s) = 1 - \frac{(1-2v)}{2\mu(1-v)k_z}\frac{s\left(\eta^2+\xi^2\right)}{\left(\eta^2-\xi^2\right)^2}$.

A.2. Elements of transfer matrix S

$$S_{11} = S_{22} = ch\xi z, \qquad \mu\xi S_{12} = \frac{S_{21}}{\mu\xi} = sh\xi z.$$

Appendix B. Formulation of matrices φ_{n+1} and ψ_{n+1} for layer $n+1$

For layer $n+1$ with $z > Z_n$, by using forward transfer matrix method, we obtain vectors $\overline{\mathbf{X}}_m(z)$ and $\overline{\mathbf{Y}}_m(z)$ as

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{pmatrix}_{n+1,z-Z_n} \begin{Bmatrix} \bar{u}_{vm} \\ \bar{\sigma}_{zm} \\ \bar{p}_m \\ \bar{w}_m \\ \bar{\tau}_{vzm} \\ \bar{v}_{zm} \end{Bmatrix}_{z=Z_n} = \begin{Bmatrix} \bar{u}_{vm} \\ \bar{\sigma}_{zm} \\ \bar{p}_m \\ \bar{w}_m \\ \bar{\tau}_{vzm} \\ \bar{v}_{zm} \end{Bmatrix}_{z>Z_n}, \tag{B.1a}$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}_{n+1,z-Z_n} \begin{Bmatrix} \bar{u}_{hm} \\ \bar{\tau}_{hzm} \end{Bmatrix}_{z=Z_n} = \begin{Bmatrix} \bar{u}_{hm} \\ \bar{\tau}_{hzm} \end{Bmatrix}_{z>Z_n}. \tag{B.1b}$$

The natural regularity conditions require the displacement and water flow velocity should reduce to zero and the stresses and pore water pressure should be bounded as $z \to +\infty$. Then we get

$$\left\{ \begin{array}{c} \bar{u}_{vm} \\ \bar{w}_m \\ \bar{v}_{zm} \end{array} \right\}_{z=Z_n} = -\lim_{z\to\infty} \begin{pmatrix} T_{11} & T_{14} & T_{16} \\ T_{41} & T_{44} & T_{46} \\ T_{61} & T_{64} & T_{66} \end{pmatrix}^{-1}_{n+1,z-Z_n} \begin{pmatrix} T_{12} & T_{13} & T_{15} \\ T_{42} & T_{43} & T_{45} \\ T_{62} & T_{63} & T_{65} \end{pmatrix}_{n+1,z-Z_n} \left\{ \begin{array}{c} \bar{\sigma}_{zm} \\ \bar{p}_m \\ \bar{\tau}_{vzm} \end{array} \right\}_{z=Z_n}, \tag{B.2a}$$

$$\lim_{z\to\infty} \begin{pmatrix} S_{11} & S_{12} \end{pmatrix}_{n+1,z-Z_n} \left\{ \begin{array}{c} \bar{u}_{hm} \\ \bar{\tau}_{hzm} \end{array} \right\}_{z=Z_n} = 0. \tag{B.2b}$$

From Eqs. (B.2a) and (B.2b), we get

$$\boldsymbol{\varphi}_{n+1}\overline{\mathbf{X}}(Z_n) = \mathbf{0}, \tag{B.3a}$$

$$\boldsymbol{\psi}_{n+1}\overline{\mathbf{Y}}(Z_n) = \mathbf{0}, \tag{B.3b}$$

where

$$\boldsymbol{\varphi}_{n+1} = \frac{1}{\Delta_1 - 1} \begin{pmatrix} 1 & \frac{\Delta_1}{2\mu_{n+1}\xi} & \frac{\Delta_2}{2\mu_{n+1}\xi} & 0 & -\frac{1}{2\mu_{n+1}\xi} & 0 \\ 0 & -\frac{1}{2\mu_{n+1}\xi} & -\frac{\Delta_2}{2\mu_{n+1}\xi} & 1 & \frac{\Delta_1}{2\mu_{n+1}\xi} & 0 \\ 0 & s\frac{\Delta_2}{2\mu_{n+1}\xi} & (1-\Delta_3)k_{z,n+1}\eta_{n+1} & 0 & -s\frac{\Delta_2}{2\mu_{n+1}\xi} & 1 \end{pmatrix}, \tag{B.4a}$$

$$\boldsymbol{\psi}_{n+1} = \left(1, \frac{1}{\mu_{n+1}\xi}\right), \tag{B.4b}$$

and $\Delta_1 = \frac{\delta_{n+1}}{4}\left(2\phi_{3,n+1} + \frac{\eta_{n+1}}{\xi}(\phi_{2,n+1} - \phi_{3,n+1})\right)$, $\Delta_2 = \delta_{n+1}\xi\phi_{0,n+1}\left[\frac{\eta_{n+1}}{\xi} - 1\right]$, $\Delta_3 = \frac{\delta_{n+1}}{4}\left(2\phi_{2,n+1} - \frac{\xi}{\eta_{n+1}}(\phi_{2,n+1} - \phi_{3,n+1})\right)$.

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